

# Experimental study of the oscillations of a rotating drop

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Two- and three-lobed oscillations of a rotating liquid drop immersed in an immiscible fluid of comparable density and the same angular velocity were studied experimentally. Using acoustically suspended drops, it has been found that the relative change in the resonance frequencies of the axisymmetric drop-shape oscillations  $\Delta\omega_l/\omega_l^{(0)}$  is proportional to the square of the normalized angular speed  $(\Omega/\omega_l^{(0)})^2$  when  $\omega_l > 2\Omega$ . This is in agreement with a recent analytical study of the same problem. Some preliminary results regarding the effect of rotation on the free-decay rate of the two-lobed oscillations are also presented.

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## 1. Introduction

In spite of being one of the basic problems in fluid mechanics, the dynamics of the oscillations of rotating liquid drops have generated little interest except as simple models for systems much larger than drops (stars) or much smaller (atomic nuclei). The advent of fully equipped, manned scientific laboratories placed in the greatly reduced gravity environment of Earth-orbit has, however, made experimentally available an unconfined liquid drop held together only by surface tension. Both the oscillatory and rotational dynamics of such unconfined liquid drops will be investigated in a forthcoming Spacelab experiment, using drops that are acoustically positioned and manipulated in air. It was in preparation for this space experiment that the present Earthbound study using immiscible liquid systems was initiated.

Ground-based experiments dealing only with rotation (Wang *et al.* 1982) and oscillation (Trinh, Zwern & Wang 1982, and Trinh & Wang 1982) have already been carried out. This paper reports the outcome of a series of measurements for the vibrational parameters of rotating drops. The quantitative determination of the effects of rotation upon both the resonance frequencies and the decay rates of the first modes of the drop-shape oscillations is of principal interest. Parts of the experimental results are favourably compared with the predictions of a recent analytical study of the same problem (Busse 1984).

A summary of the relevant theoretical results is presented in §2. A cursory description of the experimental technique and the apparatus is given in §3. Section 4 closes the paper with a description and discussion of the experimental observations.

## 2. Theoretical background

Busse (1984) investigated the effects of rotation on the frequencies of the shape oscillations of a drop immersed in a fluid of the same or different density, moving at the same fixed angular velocity. He considered the linear, incompressible, inviscid,

and time-dependent theory of rotating flows. Additional assumptions were that the rotation-induced shape deformation remained small and axisymmetric, that the amplitude of the oscillations were small, and that no inertial waves were excited in the host fluid. This approach yielded analytical expressions for the effect of rotation on the resonance frequencies of both the axisymmetric and non-axisymmetric modes of shape oscillation. The results obtained for axisymmetric oscillations are of particular interest in this paper and can be written as

$$\frac{\Delta\omega_l}{\omega_l^{(0)}} = \frac{\omega_l - \omega_l^{(0)}}{\omega_l^{(0)}} = \frac{r_0^3 \Omega^2}{T} (A + B) \quad \text{for } \omega_l > 2\Omega, \quad (1)$$

$$A = 2 \left[ \frac{\rho^i}{l(l+2)(2l-1)} + \frac{\rho^o}{(l^2-1)(2l+3)} \right], \quad (2)$$

$$B = \frac{\rho^i - \rho^o}{6(2l-1)(2l+3)} \left[ \frac{l^4 + 2l^3 - 4l^2 - 5l + 6}{(l+2)(l-1)} - \frac{l^2(l+4)\rho^o + (l+1)^2(l-3)\rho^i}{4[\rho^i(l+1) + l\rho^o]} \right], \quad (3)$$

$$\omega_l^{(0)} = \left[ \frac{l(l+1)(l-1)(l+2)T}{r_0^3[(l+1)\rho^i + l\rho^o]} \right]^{\frac{1}{2}}, \quad (4)$$

where  $\omega$  is the resonance frequency,  $r_0$  is the equivalent spherical radius of the drop,  $\Omega$  is the angular frequency of rotation,  $T$  is the interfacial tension, and  $\rho$  is the fluid density. The subscript  $l$  denotes the mode of oscillation, and the superscripts (0), i, and o refer to the state of non-rotation, the inside (drop) fluid, and the outside fluid, respectively.

The characteristics of the governing equation from which (1) was obtained change from elliptic to hyperbolic at  $\omega_l = 2\Omega$ . Equation (1) is valid only for  $\omega_l > 2\Omega$ . For  $\omega_l \leq 2\Omega$ , inertial waves could be excited in the outside fluid medium (Greenspan 1969). Equation (2) expresses the effects owing to the Coriolis acceleration which provides an effective restoring force even when  $\rho^i = \rho^o$ . The terms in (3) quantify the effects of the centrifugal force arising from the density difference. The dependence of the normalized shift in resonance frequency is evident for the interfacial tension ( $\propto T^{-1}$ ), the volume ( $\propto r_0^3$ ) and the angular velocity ( $\propto \Omega^2$ ). For  $\rho^i > \rho^o$ ,  $\Delta\omega_l$  always remains positive; but for  $\rho^i < \rho^o$ ,  $\Delta\omega_l$  could become negative depending on the values of  $\rho^i$  and  $l$ . The expression given in (4) for the resonance frequency of a non-rotating drop was first obtained by Lamb (1932).

### 3. Experimental apparatus and technique

#### 3.1. The apparatus

An acoustic-levitation technique, already described in a previous study (Trinh & Wang 1982), has been modified to incorporate a rotational capability. The acoustic-radiation forces generated in a liquid-filled resonant cavity were used to both position and deform the liquid drops. In this particular case, the cell was cylindrical, with the vertical axis coinciding with the rotation axis. The liquid cell was placed on a turntable and the whole apparatus was rotated while the drops were levitated or trapped at the rotation axis. Measurements were taken after reaching the steady-state rotation phase when the drop angular velocity matched that of the host liquid and the solid cylindrical container.

Figure 1 is a schematic representation of the experimental system. As mentioned above, the heart of the apparatus was a transparent lucite cylindrical cell ( $r = 4.7$  cm,  $h = 7.4$  cm), fitted on a precision-machined turntable, and directly coupled to a

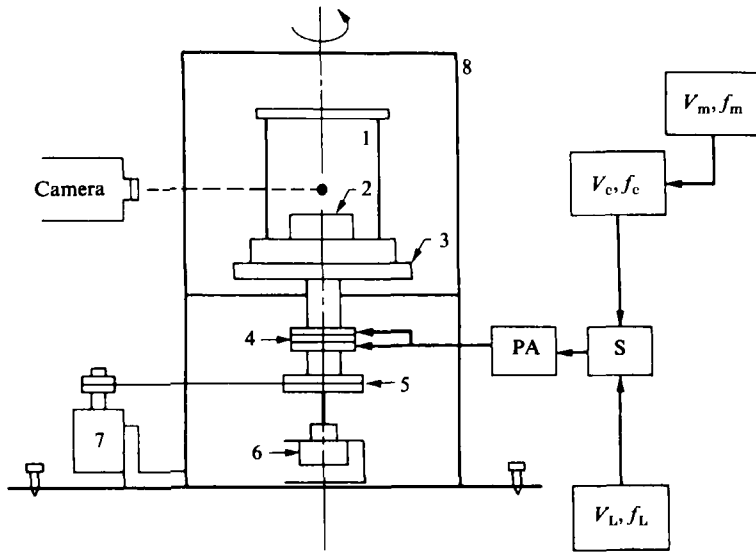


FIGURE 1. Schematic of the experimental set-up. 1, Cylindrical acoustic cell; 2, aluminium block attached to a piezoelectric transducer; 3, turntable; 4, slip rings for electrical contact; 5, belt drive; 6, light encoder; 7, drive motor; 8, support frame; PA, power amplifier, and S, summer.

piezoelectric transducer. The liquid-filled chamber was closed by a rigid top, firmly attached to the cylinder after all visible bubbles were removed. In order to avoid cavitation, the host liquid (distilled water) was thoroughly outgassed and freed of suspended solid impurities. The angular speed was measured with the light encoder fixed to the rotating shaft.

The two ultrasonic standing waves used to trap and to distort the suspended drop had frequencies of 42 and 92 kHz, respectively. For the largest drop size used in this experiment, the  $Kr_0$  values were 0.92 and 2.01, where  $r_0$  is the drop radius and  $K = 2\pi/\lambda$ ,  $\lambda$  being the acoustic wavelength of a plane wave in the host fluid. The  $Kr_0$  parameter is a measure of acoustic radiation coupling to the static shape of the drop.

Because of the additional complications introduced by the cylindrical geometry and rotational dynamics, previously developed optical measurement techniques could not be used. Consequently, the data gathering was performed visually, as well as through video and moderately high-speed cinéfilm records.

### 3.2. The method

The drops were driven into oscillations through modulation of the acoustic forces deforming the shape of the fluid sphere. The voltage imposed across the transducer was approximately expressed by

$$V = V_p \sin(2\pi f_p t) + V_c \sin(2\pi f_c t) \cos(2\pi f_m t), \quad (5)$$

where  $V_p$  drove the standing-wave mode used to position the drop ( $f_p = 42$  kHz),  $V_c$  excited the mode used to induce drop-shape oscillations ( $f_c = 92$  kHz), and  $f_m$  was the low modulation frequency.

The shape oscillations of the liquid drop were closely observed using a video system. The resonance frequencies of the drop corresponding to the  $l = 2$  and  $l = 3$  modes

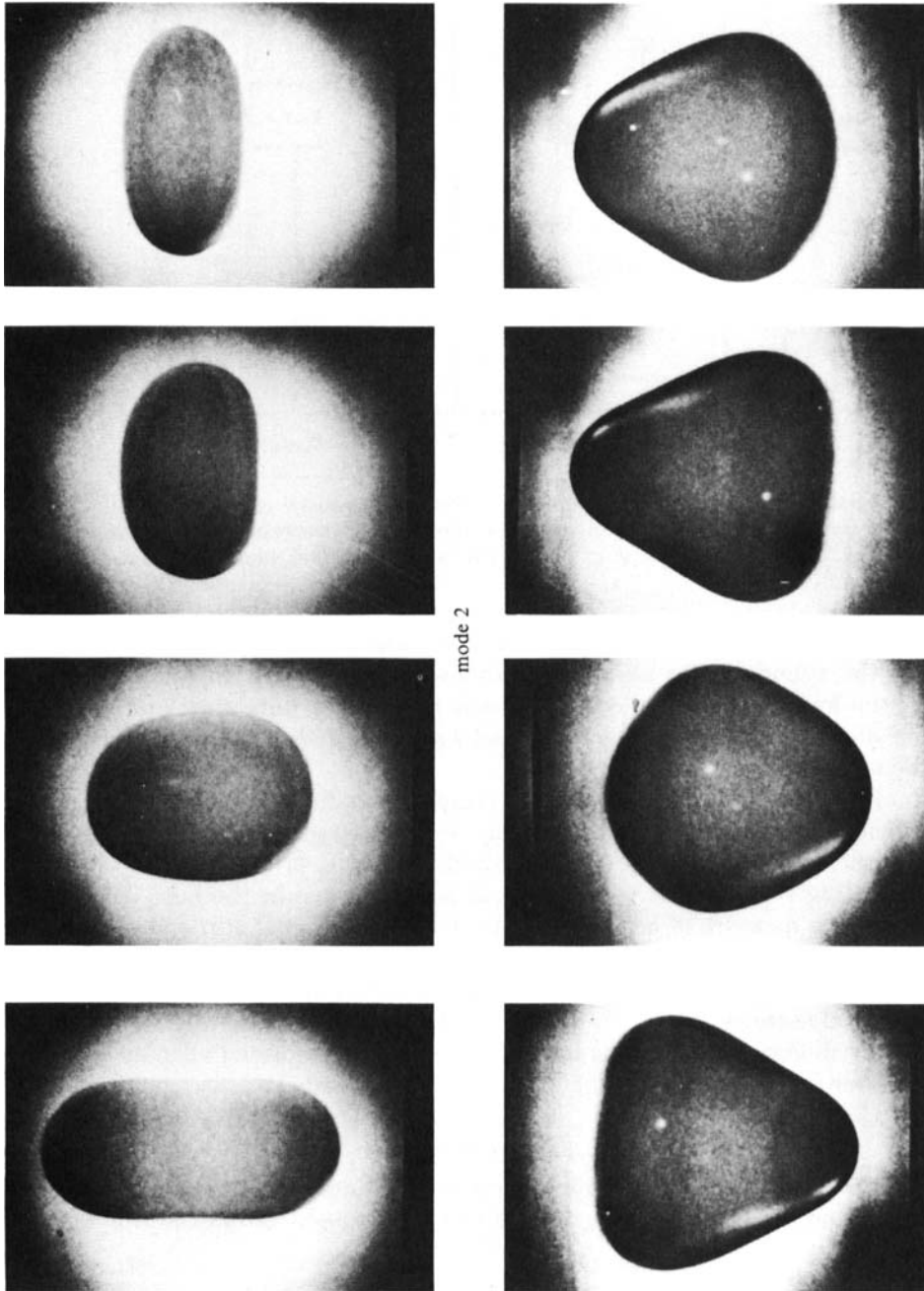


FIGURE 2. Axisymmetric oscillations of drops oscillating in the  $l = 2$  and  $l = 3$  modes (adapted from Trinh *et al.* 1982). The amplitudes of the oscillations are much larger in the figure for the purpose of illustration than actually used in this work.

were determined by slowly sweeping the low-frequency signal ( $f_m$ ), and by locating the frequency of maximum response.

The resonant-mode identification was made through stroboscopic illumination, thus allowing an apparent slowing down of the oscillations. The shapes of drops oscillating in the  $l = 2$  and  $l = 3$  modes are illustrated in figure 2.

According to a linear theory (Marston 1980), the response of a non-rotating drop to the acoustic fields may be approximated by an expansion

$$r(\theta, t) = r_0 + x(\theta, t), \tag{6}$$

where

$$x(\theta, t) = \sum_{l=2}^{\infty} [x_{pl}^{\text{stat}} + x_l^{\text{stat}} + x_l \cos(2\pi f_m t + \Phi_l)] P_l(\cos \theta) \tag{7}$$

and  $x_{pl}^{\text{stat}}$  is the  $l$ th component of the static distortion due to the positioning acoustic field,  $x_l^{\text{stat}}$  is the static distortion due to the deforming field, and  $x_l$  expresses the oscillating part due to the deforming field. For a rotating drop one may assume that  $x(\theta, t)$  again denoted the acoustically induced deformation. However, the effects of rotation on the equilibrium shape should be included by replacing  $r_0$  in (6) by an appropriate expression  $r(\theta)$ .

We assumed that, when  $f_m = \frac{1}{2}f_2$ ,

$$x_2 \gg x_{pl}^{\text{stat}} \gg x_{l+2}, \tag{8}$$

and that, when  $f_m = \frac{1}{2}f_3$ ,

$$x_3 \gg x_{pl}^{\text{stat}} \gg x_{l+3}. \tag{9}$$

The value of  $x_l^{\text{stat}}$  was not negligible, as in the earlier experimental work (Trinh *et al.* 1982). A less sensitive detection technique, used in the present study, required an increase in the magnitude of  $x_l$ , leading to an increased magnitude of  $x_l^{\text{stat}}$ . Moreover, as the angular speed  $\Omega$  became significant, an increase in  $x_l$  was necessary to excite the oscillations.

The decay process was studied by first driving the drops into oscillation at the fundamental resonance frequency ( $f_m = \frac{1}{2}f_2$ ), and then allowing them to decay freely. The oscillatory dissipation was recorded on high-speed cinéfilm, and the decay constant was determined from these records. Using the same notation, the free-decay process of a non-rotating drop may be approximately described by

$$x(\theta, t > 0) = \sum_{l=2}^{\infty} [x_l^{\text{stat}} \cos(2\pi f_l' t + \Phi_l') \exp(b_l' t) + x_l \cos(2\pi f_l'' t + \Phi_l'') \exp(b_l'' t)] P_l \cos \theta, \tag{10}$$

where the prime and doubled prime refer to the decay of the original static and oscillatory distortions respectively, and  $b_l'$  and  $b_l''$  are the decay constants associated with  $x_l^{\text{stat}}$  and  $x_l$  respectively. Even if we assume a similar expression for the free decay of a rotating drop, a complicated decay process should be anticipated since  $x_l^{\text{stat}}$  was not negligible in the present study.

## 4. Experimental results and observations

### 4.1. Resonance frequencies determination

Four different drop volumes, 0.55 ml ( $r_0 = 0.51$  cm), 0.40 ml ( $r_0 = 0.46$  cm), 0.30 ml ( $r_0 = 0.41$  cm), and 0.20 ml ( $r_0 = 0.36$  cm), were chosen for the measurement of the resonance frequencies. The density difference between the drop and the host liquids

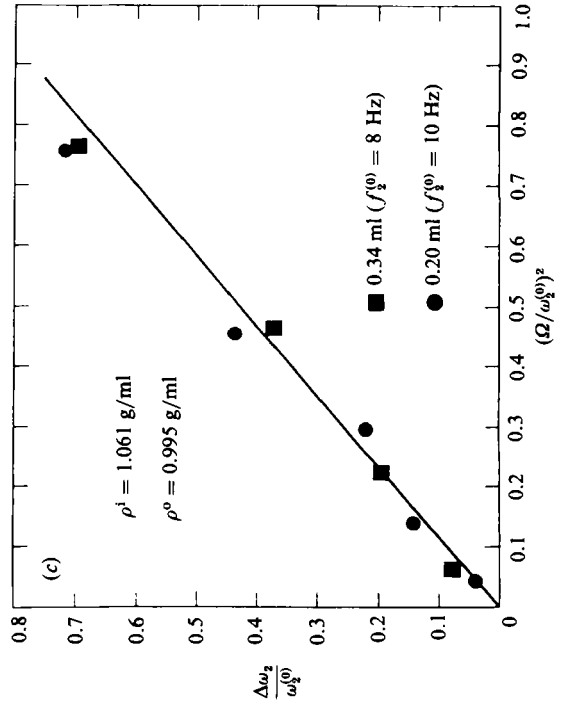
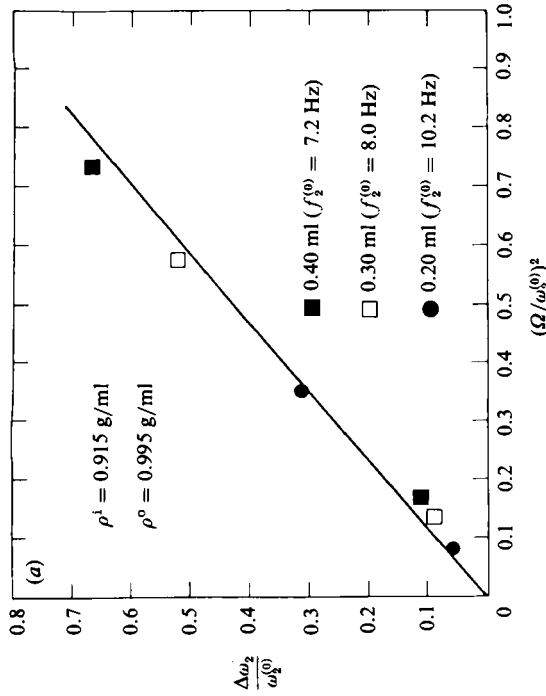
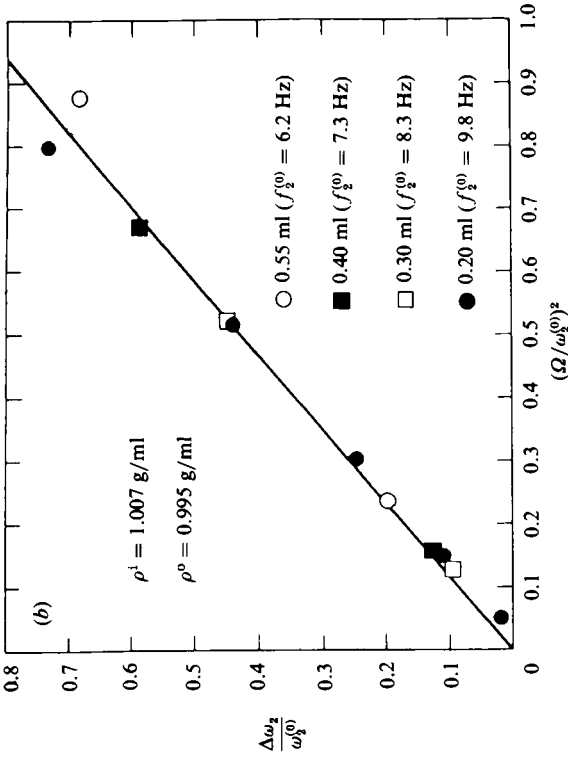


FIGURE 3. The relative shift in the resonance frequency as a function of the square of the normalized rotation rate for  $l = 2$  and (a)  $\rho^1 < \rho^0$ , (b)  $\rho^1 \approx \rho^0$ , and (c)  $\rho^1 > \rho^0$ . The solid lines were obtained from equation (11).

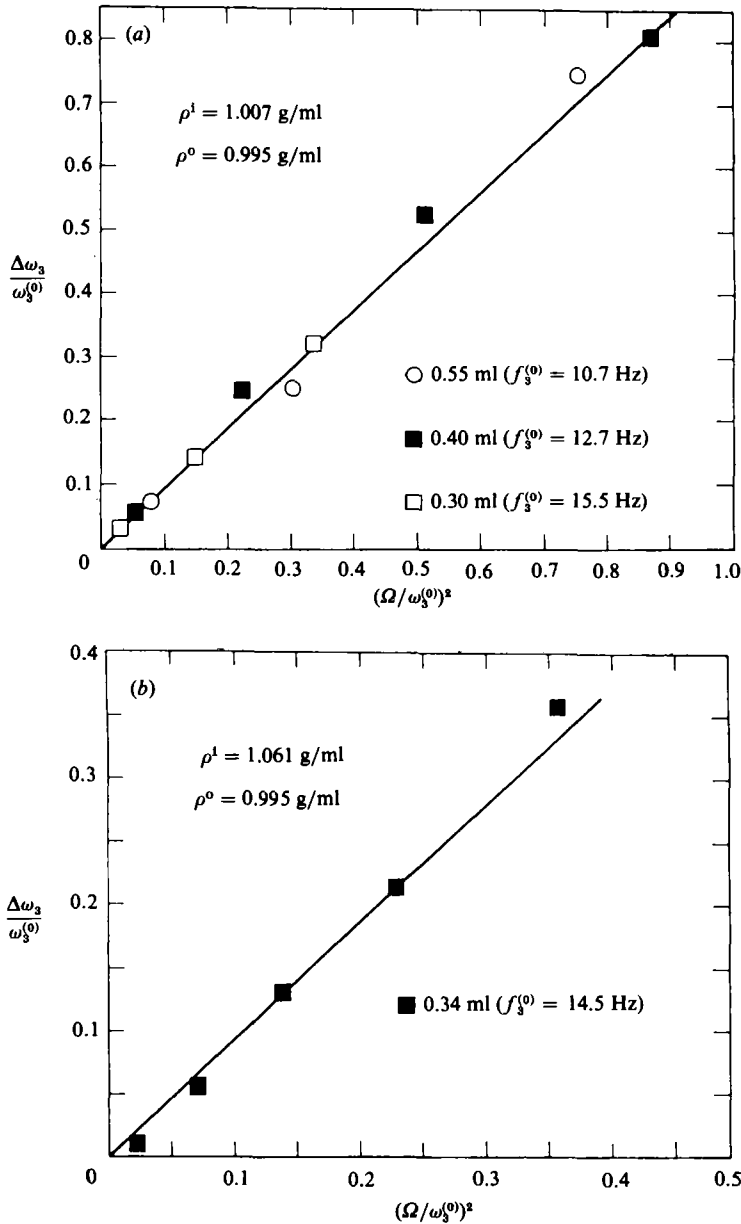


FIGURE 4. The relative shift in the resonance frequency as a function of the square of the normalized rotation rate for  $l = 3$  and (a)  $\rho^l \approx \rho^o$  and (b)  $\rho^l > \rho^o$ . The solid lines were obtained from (11).

was varied from  $-8$ – $+6\%$  of the host density. The drop liquid was a mixture of low-viscosity silicone oil ( $\nu < 10 \text{ cSt}$ ) and carbon tetrachloride in varying concentrations in order to adjust the density.

Oscillation amplitudes up to  $20\%$  drop radius were used for the measurements and were determined at the rotation axis in the vertical direction where optical distortion was at a minimum. The typical uncertainty in the frequency measurements was  $\pm 0.1 \text{ Hz}$ , or about  $2\%$ . The angular speed was observed to remain within  $\pm 0.05 \text{ rev/s}$  of the set value.

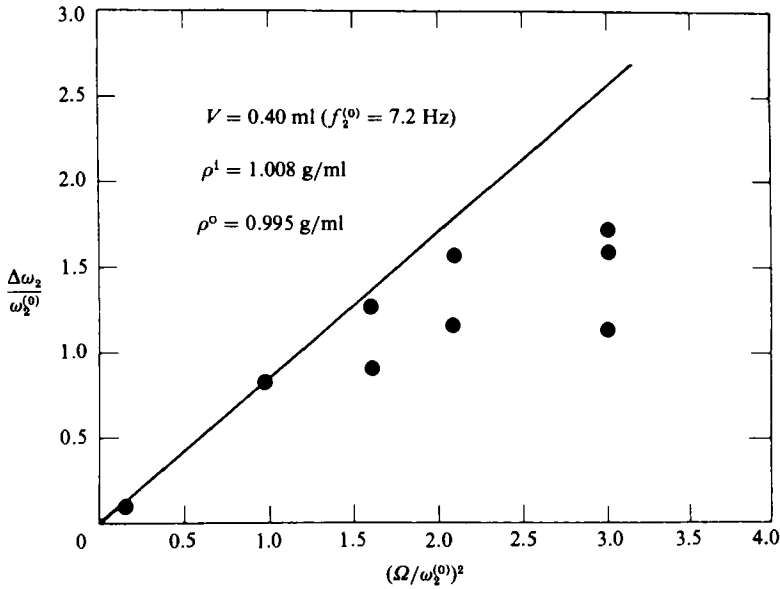


FIGURE 5. Observed multiplicity in the resonance frequency at large rotation rates ( $l = 2$ ).

Generally, a sequence of measurements was taken by alternating readings in a non-rotating state with those at a determined rotation velocity in order to monitor any time dependence of the interfacial tension. Such changes may be brought about by the slow rise in temperature ( $\sim 4^\circ\text{C}$ ) within the cell due to acoustic heating taking place throughout the duration of a set of measurements (typical experiment duration was 2 h).

To eliminate the problem arising from an inconstant interfacial tension, it was decided to plot the data by using a reduced variable for the rotation velocity. The non-dimensional variable chosen was  $\beta_l = (\Omega/\omega_l^{(0)})^2$ . Using (4) to replace  $T$  in (1), one gets the following equation for the relative shift in the resonance frequencies:

$$\frac{\Delta\omega_l}{\omega_l^{(0)}} = \left(\frac{\Omega}{\omega_l^{(0)}}\right)^2 \frac{l(l+1)(l-1)(l+2)}{(l+1)\rho^l + l\rho^0} [A + B]. \quad (11)$$

In practice,  $\omega_l^{(0)}$  was obtained experimentally by measuring the resonance frequencies of non-rotating drops. This is allowable because of the low viscosity of the fluids used in the experiments.

Figure 3(a)–(c) display some of the experimental results for the fundamental mode and for  $\rho^l < \rho^0$ ,  $\rho^l \approx \rho^0$ , and  $\rho^l > \rho^0$ , respectively. The experimental uncertainty was within the size of the symbols used in these figures. Figure 4(a) and (b) display data for the  $l = 3$  mode and for  $\rho^l \approx \rho^0$  and  $\rho^l > \rho^0$ , respectively. The experimental results for the  $\rho^l > \rho^0$  case do not extend to large values of the normalized angular speed owing to wobbling induced by a less-than-perfect alignment of the cylinder, drop, and rotation axes. The solid lines shown in these figures were obtained from (11).

The slopes of the solid lines in figure 3 are about 0.85 and those in figure 4 about 0.93. They reveal that the effect of rotation was felt more strongly in the three-lobed than in the two-lobed oscillations. They also show that, in the range of density differences studied (i.e.  $-8$  to  $+6\%$  of the host-liquid density), the resonance-frequency shift was not significantly affected by the density of the drop.



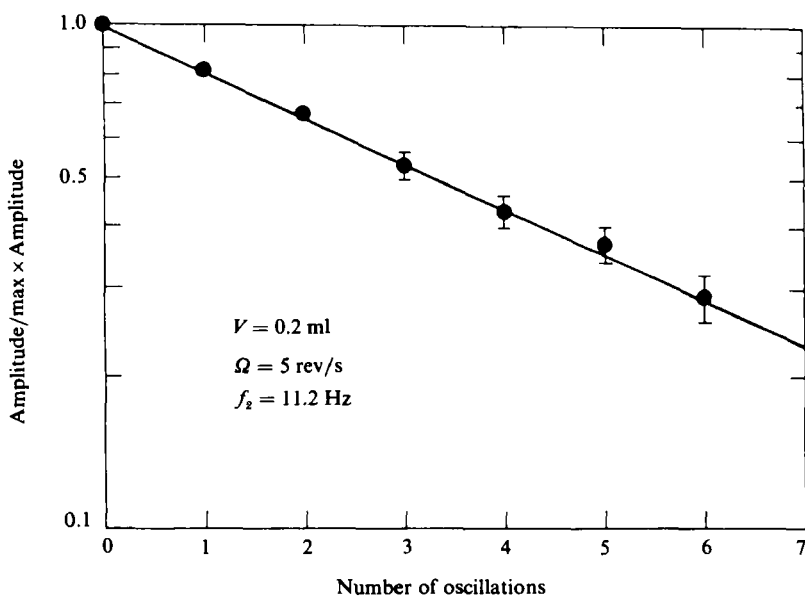


FIGURE 6. The decay process of a 0.20 ml drop at 5 rev/s ( $l = 2$ ).

A close agreement of the data with the theoretical predictions verifies the linear dependence of the relative frequency shift on the square of the normalized angular speed. The oscillation amplitudes used in the experiments were not infinitesimal as was assumed in the theory. The resonance frequency is known to decrease with an increasing oscillation amplitude (Trinh & Wang 1982). Such a decrease, owing to finite amplitude excited in the experiments, is estimated to be about 3–5%.

An intriguing experimental observation was made as the angular speed of rotation was increased, i.e. for  $(\Omega/\omega_l^{(0)})^2 > 1.0$ . Evidence was found for the existence of more than one resonance frequency for the  $l = 2$  mode for a given rotational velocity. Those results are displayed in figure 5. Stroboscopic illumination revealed that they were all oscillations of the oblate–prolate type. One must note that this rotational velocity limit corresponds to  $\omega_l \leq 2\Omega$ , where the assumptions of the linear theory discussed above break down, and inertial waves could be excited in the chamber. The present experimental apparatus, however, does not readily lend itself to the observation of such wave phenomena.

#### 4.2. Decay-constant measurement

The damping-constant measurements were carried out for the  $l = 2$  mode by recording the decay process of an initially oscillating drop on high-speed cinéfilm, and by extracting the relevant time constant from a semi-logarithmic plot of amplitude *versus* time. The use of this technique implies an exponential-decay process with a single-time constant, and has proved reliable in the study of non-rotating drops. In the case of rotating drops, this method only yielded damping-constant values for small drop volumes ( $V \leq 0.20$  ml) at low rotation velocities. Figure 6 reproduces experimental data obtained for a 0.20 ml drop rotating at 5 rev/s, initially driven into oscillation at a resonance frequency of 11.2 Hz. Figure 7 displays measured damping constants  $b_2''$  at various rotation velocities for the same 0.20 ml drop. A slight increasing trend may readily be observed.

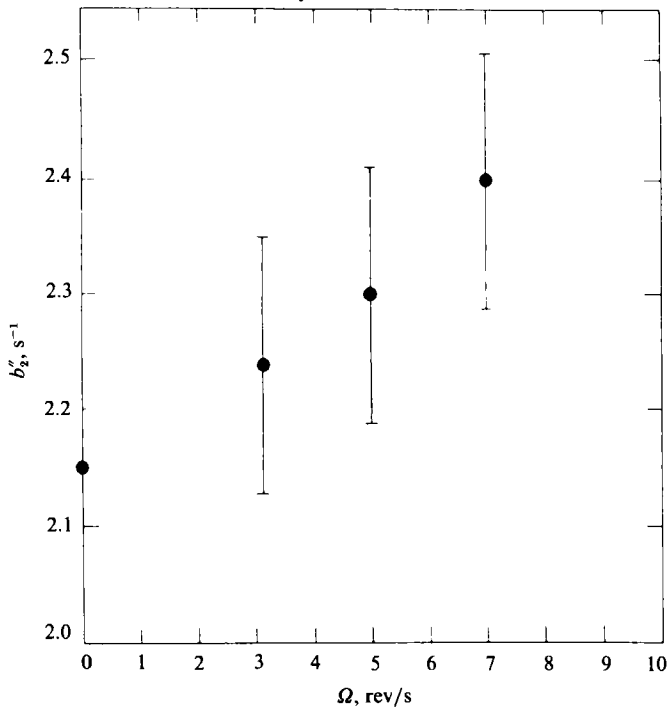


FIGURE 7. The damping constant as a function of the rotation rate for a 0.20 ml drop ( $l = 2$ ).

For larger drops ( $V \geq 0.40$  ml), the observed decay process is no longer characterized by a simple exponential curve for rotation rates  $\Omega > 3$  rev/s. Typical results are reproduced in figure 8 for the free decay of a 0.40 ml drop at 3 and 7 rev/s. The data obtained at the higher rotation rate reflect a sharp initial decrease in the amplitude during the first cycle, and a subsequently significantly slower decay process. A check on the behaviour of the 0.20 ml drop showed a similar trend, but at 9 rev/s ( $f_2 = 16$  Hz). As discussed before, this complex decay process could be owing to the non-negligible static distortion  $x_i^{\text{stat}}$  (see (10)). One might again note that the onset of this discontinuity roughly coincides with the reaching of the limit  $\omega_l \approx 2\Omega$ .

#### 4.3. Discussion and conclusion

Satisfactory agreement was obtained between the experimental data and the theoretical predictions for  $\omega_l > 2\Omega$ . The relative shift in resonance frequency  $\Delta\omega_l/\omega_l^{(0)}$  was found to be linearly dependent on the square of the relative rotation velocity  $(\Omega/\omega_l^{(0)})^2$  for a given set of densities  $\rho^i$  and  $\rho^o$ . Observed deviations at higher rotation rates and the multiplicity of the resonant modes could not be explained by the available theory.

The preliminary results of decay-constant measurements indicate an increase in the free-decay rate for small values of  $\Omega$  and a more complex decay process for large values of  $\Omega$ . The interpretation of the data remains ambiguous, however, owing to the undetermined role of the acoustic-radiation pressure coupling to the static shape of the drop. Such a coupling would become more significant as the  $Kr_0$  parameter increases, i.e. when the drop volume becomes larger.

Conjectures about the possible influence of inertial waves on the resonance frequencies and the decay rates for  $\omega_l \leq 2\Omega$  could be made, though the physical

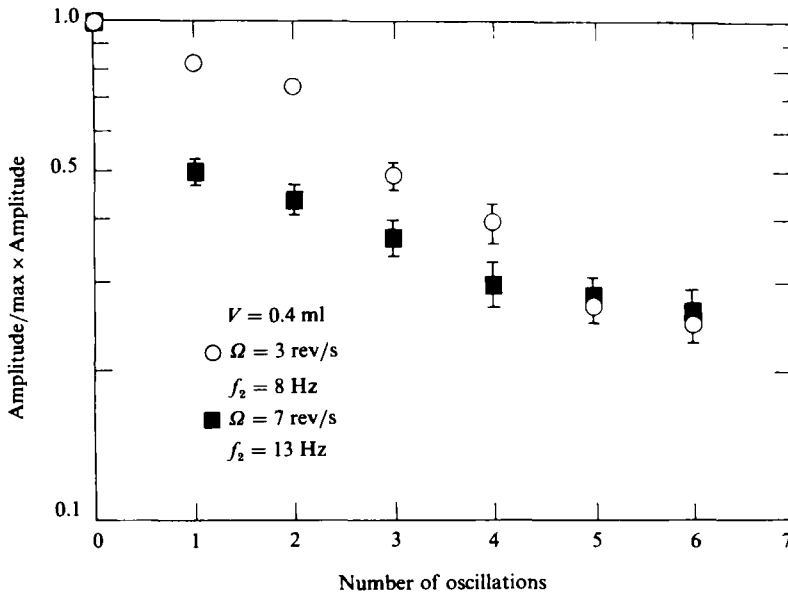


FIGURE 8. The decay process of a 0.40 ml drop at 3 and 7 rev/s ( $l = 2$ ).

mechanism through which these waves might couple to the drop-shape oscillations is not clear. Although of fundamental interest, the treatment of this phenomenon is beyond the scope of the present work. The result of this work, however, provides ample motivation for a detailed experimental study of the interaction of inertial waves with drop-shape oscillations, as well as a theoretical investigation on the decay process of rotating and oscillating liquid drops.

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#### REFERENCES

- BUSSE, F. H. 1984 *J. Fluid Mech.* **142**, 1.  
 GREENSPAN, H. P. 1969 *The Theory of Rotating Fluids*, p. 10 and p. 51. Cambridge University Press.  
 LAMB, H. 1932 *Hydrodynamics*, pp. 473–639. Cambridge University Press.  
 MARSTON, P. L. 1980 *J. Acoust. Soc. Am.* **67**, 15.  
 TRINH, E. & WANG, T. 1982 *J. Fluid Mech.* **122**, 315.  
 TRINH, E., ZWERN, A. & WANG, T. 1982 *J. Fluid Mech.* **115**, 453.  
 WANG, T. G., TAGG, R., CAMMACK, L. & CROONQUIST, A. 1982 *Proc. 2nd Int. Colloquium on Drops and Bubbles, JPL Tech. Rep. 82-7*, pp. 203–213. Jet Propulsion Laboratory, Pasadena, California.